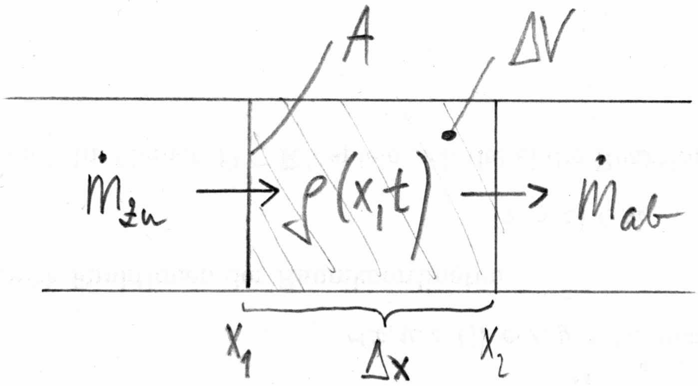
Die Kontinuitätsgleichung



$\Delta x : I_x = [x_1, x_2]$

$\Delta t : I_t = [t_1, t_2]$

<u>Während dt:</u> $dm_{zu} = \rho(x_1,t)\cdot A \cdot v(x_1,t)\,dt$, $dm_{ab} = \rho(x_2,t)\,A\,v(x_2,t)\,dt$

$$\dot m_{zu} = \dot m(x_1,t) = \rho(x_1,t)\cdot A\cdot v(x_1,t), \quad \dot m_{ab} = \dot m(x_2,t) = \rho(x_2,t)\cdot A\cdot v(x_2,t)$$

$$dm = dm_{zu} - dm_{ab} \Rightarrow \text{in } [t_1,t_2]\ \underline{\Delta m} = \int_{t_1}^{t_2} dm = \int_{t_1}^{t_2}(dm_{zu} - dm_{ab})$$

$$dm = \left(\rho(x,t_2) - \rho(x,t_1)\right)\cdot A\cdot dx \ \ldots$$ Massenänderung im Intervall $[t_1,t_2]$ im Volumen $dV = A\cdot dx$

$\downarrow \int_{x_1}^{x_2}$

$$\Delta V: \begin{cases} \Delta m = \displaystyle\int_{x_1}^{x_2}\left(\rho(x,t_2) - \rho(x,t_1)\right)\cdot \overbrace{A\cdot dx}^{=dV} \quad \text{im Intervall } [t_1,t_2] \\ \underline{\underline{\Delta m}} = \displaystyle\int_{t_1}^{t_2}(\dot m_{zu} - \dot m_{ab})\,dt = \left(\int_{t_1}^{t_2}(dm_{zu} - dm_{ab}) = \right) \end{cases}$$

im Zeitintervall $[t_1,t_2]$

$$= \int_{t_1}^{t_2}\left(\dot m(x_1,t) - \dot m(x_2,t)\right)dt = \int_{t_1}^{t_2}\left(\rho(x_1,t)\cdot v(x_1,t) - \rho(x_2,t)\cdot v(x_2,t)\right)\cdot A\cdot dt$$

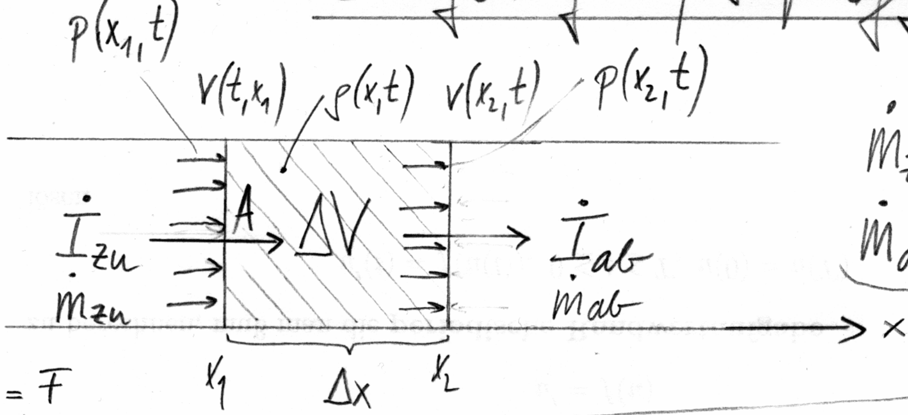
$$\underbrace{\int_{x_1}^{x_2}\left(\rho(x,t_2) - \rho(x,t_1)\right)}_{=\int_{t_1}^{t_2}\frac{\partial\rho(x,t)}{\partial t}\,dt} A\cdot dx = -\int_{t_1}^{t_2}\underbrace{\left(\rho(x_2,t)\cdot v(x_2,t) - \rho(x_1,t)\,v(x_1,t)\right)}_{=\int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\rho(x,t)\cdot v(x,t)\right)dx}\cdot A\,dt$$

$\boxed{\text{HS der Diff- \& Int-Rechnung!}}$

$$\Rightarrow \int_{x_1}^{x_2}\int_{t_1}^{t_2}\left(\frac{\partial\rho(x,t)}{\partial t}\cdot A + \frac{\partial}{\partial x}\left(\rho(x,t)\cdot v(x,t)\right)\cdot A\right)dt\,dx = 0,$$

$$\forall [t_1,t_2] \wedge \forall [x_1,x_2] \Rightarrow \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho\cdot v) = 0$$

Eulergleichungen / Impulsglg (1D)



$$\dot m_{zu} = \rho(x_1,t)\cdot A\cdot v(x_1,t)$$
$$\dot m_{ab} = \rho(x_2,t)\cdot A\cdot v(x_2,t)$$

$$\overbrace{\dot I_{zu}}^{=F} = \dot m_{zu}\cdot v(x_1,t) + p(x_1,t)\cdot A = A\cdot\left(\rho(x_1,t)\cdot v^2(x_1,t) + p(x_1,t)\right)$$

$$\dot I_{ab} = \dot m_{ab}\cdot v(x_2,t) + p(x_2,t)\cdot A = A\cdot\left(\rho(x_2,t)\cdot v^2(x_2,t) + p(x_2,t)\right)$$

$$dI = \left(\rho(x,t_2)v(x,t_2) - \rho(x,t_1)v(x,t_1)\right)\overbrace{A\cdot dx}^{=dV}$$ ... infinitesimale Impulsänderung in $[t_1,t_2]$

In ΔV gilt: $$\Delta I = \int_{x_1}^{x_2}\left(\rho(x,t_2)v(x,t_2) - \rho(x,t_1)v(x,t_1)\right)\cdot A\cdot dx \quad \text{in } [t_1,t_2]$$

$$\Delta I = \int_{t_1}^{t_2}\left(\dot I_{zu} - \dot I_{ab}\right)dt \quad \text{in } [t_1,t_2]$$

$$\Rightarrow \int_{x_1}^{x_2}\left(\rho(x,t_2)v(x,t_2) - \rho(x,t_1)v(x,t_1)\right)A\,dx = \int_{t_1}^{t_2}\left(\rho(x_1,t)v^2(x_1,t) + p(x_1,t) - \left(\rho(x_2,t)v^2(x_2,t) + p(x_2,t)\right)\right)A\cdot dt$$

HS d. Diff- & Int.-R.

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} A\cdot\frac{\partial(\rho v)}{\partial t}\,dt\,dx = -\int_{t_1}^{t_2}\int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\rho v^2 + p\right)A\,dx\,dt$$

mit $\rho = \rho(x,t)$, $v = v(x,t)$, $p = p(x,t)$

$$\Rightarrow \int_{x_1}^{x_2}\int_{t_1}^{t_2} A\cdot\Big(\underbrace{\frac{\partial}{\partial t}(\rho\cdot v) + \frac{\partial}{\partial x}\left(\rho v^2 + p\right)}_{=0\ \forall [t_1,t_2]\ \&\ [x_1,x_2]}\Big)\,dx\,dt = 0$$

$$\boxed{\frac{\partial}{\partial t}(\rho\cdot v) + \frac{\partial}{\partial x}\left(\rho v^2 + p\right) = 0}$$

Viskosität vernachlässigt!

Weiters gilt mit Hilfe der KG:

$$\text{KG}: \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho \cdot v)}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \cdot v + \rho \frac{\partial v}{\partial x} = 0$$

$$\text{IG}: \quad \frac{\partial \rho}{\partial t} \cdot v + \rho \frac{\partial v}{\partial t} + \frac{\partial \rho}{\partial x} \cdot v^2 + \rho\, 2v \frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = 0$$

$$v \cdot \underbrace{\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \cdot v + \rho \frac{\partial v}{\partial x} \right)}_{= 0 \ (\text{KG!})} + \rho v \frac{\partial v}{\partial x} + \rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} = 0$$

$$\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = 0$$

Ansatz alternativ:

in dt: $dm_{zu} = \rho(x_1,t) \cdot \underbrace{A \cdot v(x_1,t) \cdot dt}_{= dV_1}$, $dm_{ab} = \rho(x_2,t) \cdot \underbrace{A \cdot v(x_2,t) \cdot dt}_{= dV_2}$

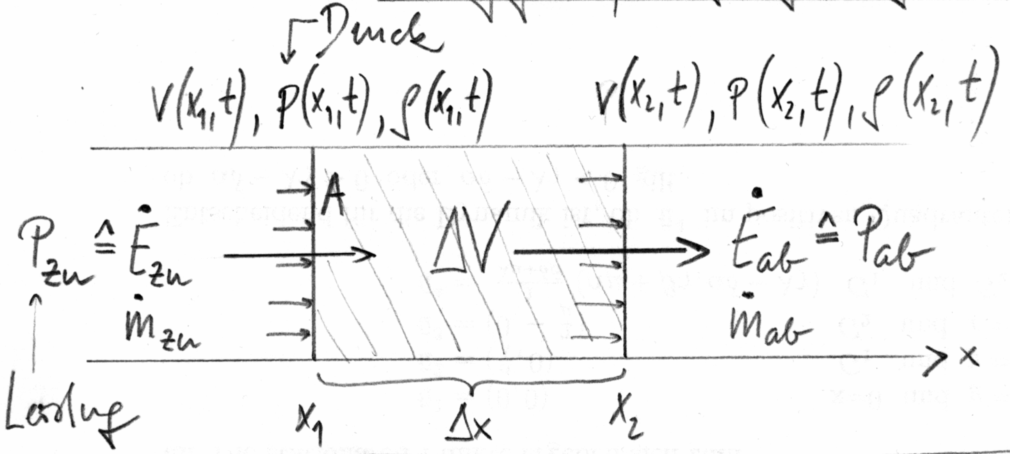
$\Rightarrow dI_{zu} = dm_{zu} \cdot v(x_1,t) + \underbrace{p(x_1,t) \cdot A \cdot dt}_{\text{Kraftstoß aufgrund des Druckes}}$, $dI_{ab} = dm_{ab} \cdot v(x_2,t) + p(x_2,t) \cdot A \cdot dt$

$\Rightarrow dI = dI_{zu} - dI_{ab}$

in $[t_1, t_2]$: $\Delta I = \int_{t_1}^{t_2} dI = \int_{t_1}^{t_2} \Big(\rho(x_1,t) v^2(x_1,t) + p(x_1,t) - \big(\rho(x_2,t) \cdot v^2(x_2,t) + p(x_2,t) \big) \Big) \cdot A \cdot dt$

Eulergleichung / Energiegleichung (1D)

⚠ $T = \text{const.}$
$\neq$ in $\dot{E}$: $c_v \cdot \dot{m} \cdot T$



$$\dot{m}_{zu} = \rho(x_1,t) \cdot A \cdot v(x_1,t)$$
$$\dot{m}_{ab} = \rho(x_2,t) \cdot A \cdot v(x_2,t)$$

$$\dot{E}_{zu} = \frac{1}{2}\dot{m}_{zu} \cdot v^2(x_1,t) + p(x_1,t) \cdot A \cdot v(x_1,t) = \frac{1}{2}\rho(x_1,t) A \cdot v^3(x_1,t) + p(x_1,t) A v(x_1,t)$$

$$\dot{E}_{ab} = \frac{1}{2}\dot{m}_{ab} \cdot v^2(x_2,t) + p(x_2,t) \cdot A \cdot v(x_2,t) = \frac{1}{2}\rho(x_2,t) \cdot A v^3(x_2,t) + p(x_2,t) \cdot A \cdot v(x_1,t)$$

infinitesimale E-Änderung in $dV = A \cdot dx$ während $[t_1,t_2]$:

$$dE = \frac{1}{2}\left[\rho(x,t_2) \cdot v^2(x,t_2) - \rho(x,t_1) \cdot v^2(x,t_1)\right] A \cdot dx$$

$\Longrightarrow$ für ΔV in $[t_1,t_2]$: $\Delta E = \int_{x_1}^{x_2} \left[\rho(x,t_2) v^2(x,t_2) - \rho(x,t_1) v^2(x,t_1)\right] \cdot A \cdot dx$

$$\Delta E = \int_{t_1}^{t_2} \left(\dot{E}_{zu} - \dot{E}_{ab}\right) dt \qquad \left(\text{in } [t_1,t_2]\right)$$

mit
$\rho = \rho(x,t)$
$v = v(x,t)$
$p = p(x,t)$

$$\Longrightarrow \int_{x_1}^{x_2} \left(\frac{1}{2}\rho(x,t_2) v^2(x,t_2) - \rho(x,t_1) v^2(x,t_1)\right) A\,dx = \int_{t_1}^{t_2} \left(\left(\frac{1}{2}\rho(x_1,t) v^3(x_1,t) + p(x_1,t) v(x_1,t)\right) - \left(\frac{1}{2}\rho(x_2,t) v^3(x_2,t) + p(x_2,t) v(x_2,t)\right)\right) A\,dt$$

HS d. Diff.-R. Int.-R.

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} \frac{1}{2} \cdot A \cdot \frac{\partial(\rho \cdot v^2)}{\partial t}\,dt\,dx = -\int_{t_1}^{t_2} A \cdot \frac{\partial}{\partial x}\left(\frac{1}{2}\rho \cdot v^3 + p \cdot v\right) dt\,dx$$

$$\Longrightarrow \int_{x_1}^{x_2}\int_{t_1}^{t_2} A \cdot \left[\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho v^3 + p \cdot v\right)\right] dt\,dx = 0$$

$$\forall [t_1,t_2]\ \&\ \forall [x_1,x_2] \Longrightarrow \frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2}\rho v^2\right)}_{=\rho_E} + \frac{\partial}{\partial x}\left(\underbrace{\frac{1}{2}\rho v^3}_{=\rho_E \cdot v} + p \cdot v\right) = 0$$

$$\Longrightarrow \frac{\partial \rho_E}{\partial t} + \frac{\partial}{\partial x}\left(v \cdot (\rho_E + p)\right) = 0$$

ρ_E ← Energiedichte