Lösung der Klein-Gordon-Gleichung (1D)

$$\ddot{\beta} = \frac{E}{\rho}\beta'' - \omega_0^2 \beta \qquad \beta = \beta(x,t), \quad \omega_0^2 = \frac{g}{l}, \quad E,\rho > 0$$

Separationsansatz: $\beta(x,t) = \theta(x)\cdot\eta(t)$

$$\beta(x,t) = \theta(x)\cdot\sin(\omega t + \varphi)$$

$$\ddot{\beta}(x,t) = -\theta(x)\cdot\omega^2\cdot\sin(\omega t+\varphi)$$

$$\beta''(x,t) = \theta''(x)\cdot\sin(\omega t+\varphi)$$

$$\rightarrow -\theta(x)\omega^2 \cancel{\sin(\omega t+\varphi)} = \frac{E}{\rho}\theta''(x)\cancel{\sin(\omega t+\varphi)} - \omega_0^2\theta(x)\cdot\cancel{\sin(\omega t+\varphi)}$$

$$\Longrightarrow \theta''(x) = \frac{\rho}{E}\cdot(\omega_0^2-\omega^2)\cdot\theta(x)$$

Fallunterscheidungen:

•) $\omega_0^2 - \omega^2 < 0$: $\theta(x) = \theta_1\cdot\sin(kx) + \theta_2\cdot\cos(kx)$

$$\theta''(x) = -\theta_1 k^2 \sin(kx) + \theta_2 k^2 \cos(kx)$$

$$\rightarrow -\theta_1 k^2\cdot\sin(kx) - \theta_2 k^2\cos(kx) = \frac{\rho}{E}(\omega_0^2-\omega^2)\cdot\left[\theta_1\sin(kx)+\theta_2\cos(kx)\right]$$

$$-k^2\cancel{\left[\theta_1\sin(kx)+\theta_2\cos(kx)\right]} = \frac{\rho}{E}(\omega_0^2-\omega^2)\cancel{\left[\theta_1\sin(kx)+\theta_2\cos(kx)\right]}$$

$$\Longrightarrow k = \underset{(-)}{\pm}\sqrt{\frac{\rho}{E}(\omega^2-\omega_0^2)}$$

$$\boxed{\beta(x,t) = \left[\theta_1\cdot\sin(kx)+\theta_2\cos(kx)\right]\cdot\sin(\omega t+\varphi)}$$ ... periodische Lösung

φ folgt aus den Anfangsbedingungen

θ_1, θ_2 folgen aus den Randbedingungen

$\cdot)\ \omega_0^2 - \omega^2 > 0$: $\theta(x) = \theta_1 \cdot e^{-kx} + \underbrace{\theta_2 e^{kx}}_{\substack{\text{math. mögl.} \\ \text{phys. nicht!}}}$ Exponentialwelle

$$\theta''(x) = \theta_1 k^2 e^{-kx} + \theta_2 k^2 e^{kx}$$

$$\Rightarrow k^2 \cancel{\left[\theta_1 e^{-kx} + \theta_2 e^{kx}\right]} = \frac{\rho}{E}\left(\omega_0^2 - \omega^2\right) \cancel{\left[\theta_1 e^{-kx} + \theta_2 \cdot e^{kx}\right]}$$

$$\Rightarrow k = (\pm)\sqrt{\frac{\rho}{E}\left(\omega_0^2 - \omega^2\right)}$$

$$\boxed{\beta(x,t) = \left[\theta_1 e^{-kx} + \theta_2 e^{kx}\right] \cdot \sin(\omega t + \varphi)}$$