Lösung von $y' = \lambda \cdot y$ mittels Reihenansatz

$$y' = \lambda \cdot y \quad , \quad y(0) = y_0$$

Ansatz: $$y = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$y' = \sum_{i=0}^{\infty} a_i \cdot i \cdot x^{i-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

$$y(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + a_3 \cdot 0^3 + \ldots = y_0$$

$$\Rightarrow \underline{a_0 = y_0}$$

$$\Rightarrow \underline{a_1} + \underline{\underline{2a_2}}\, x + \underline{\underline{\underline{3a_3}}}\, x^2 + \ldots = \lambda \cdot (\underline{y_0} + \underline{\underline{a_1}}\, x + \underline{\underline{\underline{a_2}}}\, x^2 + a_3 x^3 + \ldots)$$

$$\underline{a_1 = \lambda y_0}$$

$$2a_2 = \lambda a_1 \Rightarrow \underline{a_2} = \frac{\lambda}{2} \cdot \lambda y_0 = \underline{\frac{\lambda^2}{2} \cdot y_0}$$

$$3a_3 = \lambda a_2 \Rightarrow \underline{a_3} = \frac{\lambda}{3} \cdot a_2 = \frac{\lambda}{3} \cdot \frac{\lambda^2}{2} \cdot y_0 = \underline{\frac{\lambda^3}{6} \cdot y_0}$$

$$\Rightarrow y = y_0 + \lambda y_0 + \frac{\lambda^2}{2} \cdot y_0 + \frac{\lambda^3}{6} \cdot y_0 + \ldots$$

$$y = y_0 \underbrace{\left(1 + \lambda x + \frac{\lambda^2}{2} x^2 + \frac{\lambda^3}{6} x^3 + \ldots\right)}_{= \text{Taylor von } e^{\lambda x}} = y_0 \cdot \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} x^i$$

Lösung von $y'' = \frac{1}{y^2}$, $y(0) = y_0$, $y'(0) = 0$ mittels Reihenansatz

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 \cdot x^6 + \dots$$

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \dots$$

$$y''(x) = 2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + 30 a_6 \cdot x^4 + \dots$$

$$y(0) = a_0 = y_0$$

$$y'(0) = a_1 = 0$$

$$\Rightarrow y'' \cdot y^2 = 1$$

$$(2a_2 + 6a_3 x + 12 a_4 x^2 + \dots) \cdot \underbrace{(y_0 + a_2 x^2 + a_3 x^3 + \dots)^2}_{\text{liefert nur gerade Pot. von } x} =$$

$$= 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + \dots$$

$$2a_2 y_0^2 = 1 \Rightarrow a_2 = \frac{1}{2y_0^2}$$

$$6a_3 \cdot y_0 = 0 \Rightarrow a_3 = 0$$

$$12 a_4 \cdot y_0^2 + 2a_2 \cdot 2y_0 a_2 = 0$$

$$12 a_4 y_0^2 + 4a_2^2 \cdot y_0 = 0$$

$$12 a_4 y_0^2 + \cancel{4} \frac{1}{\cancel{4} y_0^{\cancel{4}\,3}} \cdot \cancel{y_0} = 0 \Rightarrow a_4 = -\frac{1}{12 \cdot y_0^5}$$

$$a_5 = 0$$

$$\Rightarrow y(x) = y_0 + \frac{1}{2y_0^2} x^2 - \frac{1}{12 y_0^5} \cdot x^4 + O(x^6)$$

Lösung der Pendelgleichung $\ddot{y} + \omega^2 \sin(y) = 0$ mittels Reihenansatz

<u>ABs:</u> $y(0) = y_0$, $\dot{y}(0) = 0$

$$\sin y = y - \frac{y^3}{6} + \frac{y^5}{120} - \dots \quad \text{(Taylor)}$$

<u>Ansatz:</u> $y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \dots$

$$\Longrightarrow \dot{y}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \dots$$

$$\ddot{y}(t) = 2a_2 + 6a_3 t + 12 a_4 t^2 + \dots$$

<u>ABs:</u> $y(0) = a_0 = y_0$

$\dot{y}(0) = a_1 = 0$

$$\Longrightarrow 2a_2 + 6a_3 t + 12 a_4 t^2 + \dots + \omega^2 \cdot \Big(a_0 + \underbrace{a_1 t}_{=0} + a_2 t^2 + \dots$$

$$- \frac{1}{6}\big(a_0 + \underbrace{a_1 t}_{=0} + a_2 t^2 + \dots\big)^3$$

$$+ \frac{1}{120}\big(a_0 + \underbrace{a_1 t}_{=0} + a_2 t^2 + \dots\big)^5$$

$$- \dots \Big) = 0$$

$$\Longrightarrow \cdot)\ 2a_2 + \omega^2 \cdot \underbrace{\left(y_0 - \frac{1}{6} y_0^3 + \frac{1}{120} y_0^5 + \dots \right)}_{=\text{ Taylor von } \sin(y_0)} = 0 \Rightarrow a_2 = -\omega^2 \frac{\sin(y_0)}{2}$$

$$\cdot)\ 6a_3 + \omega^2 \cdot \underset{=a_1}{0} = 0 \Longrightarrow a_3 = 0$$

$$\cdot)\ 12 a_4 + \omega^2 \cdot \left(a_2 - \frac{1}{\cancel{6}_{\,2}} \cdot \cancel{3} \underset{=y_0}{a_0^2} a_2 \right) = 0$$

$$12 a_4 + \omega^2 \cdot (-\omega^2) \cdot \frac{\sin(y_0)}{2} \cdot \left(1 - \frac{1}{2} y_0^2 \right)$$

$$\Longrightarrow a_4 = +\omega^4 \cdot \frac{\sin(y_0)}{24} \left(1 - \frac{y_0^2}{2} \right)$$

$$y(t) \doteq y_0 - \omega^2 \frac{\sin(y_0)}{2} t^2 + \omega^4 \frac{\sin(y_0)}{24} \left(1 - \frac{y_0^2}{2} \right) \cdot t^4 + O(t^6)$$

Wenn $y_0 \downarrow \Rightarrow \sin(y_0) \approx y_0$ (Taylor)

$$\Rightarrow y(t) = y_0 - \frac{y_0}{2}(\omega t)^2 + \frac{y_0}{24} \cdot \Big(1 - \underbrace{\frac{y_0^2}{2}}_{\hookrightarrow 0,\ \text{da } y_0 \downarrow \Rightarrow y_0^2 \downarrow\downarrow}\Big) \cdot (\omega t)^4 + O(t^6)$$

$$y(t) = y_0 \Big(\underbrace{1 - \frac{1}{2}(\omega t)^2 + \frac{1}{24}(\omega t)^4 + O(t^6)}_{= \text{Taylor von } \cos(\omega t)}\Big)$$

Das ist korrekt, da $y = y_0 \cdot \cos(\omega t)$ die Lösung von $\ddot{y} + \omega^2 y = 0$ mit $y(0) = y_0$ & $\dot{y}(0) = 0$ ist!

Die logistische DGL – klassische Lösung

$\dot{y} = \lambda \cdot y \cdot (1-y)$ mit $y(0) = y_0$

Trennung der Variablen:

$$\frac{dy}{dt} = \lambda \cdot y(1-y)$$

$$\frac{dy}{y(1-y)} = \lambda \cdot dt$$

Partialbruchzerlegung:

$$\frac{1}{y\cdot(1-y)} = \frac{A}{y} + \frac{B}{1-y} = \frac{A(1-y)+B\cdot y}{y\cdot(1-y)} = \frac{A+(B-A)\cdot y}{y\cdot(1-y)}$$

$$\Rightarrow A = 1 \quad \wedge \quad B-A = 0$$
$$B-1 = 0$$
$$B = 1$$

$$\frac{1}{y} + \frac{1}{1-y} = \frac{1-y}{y(1-y)} + \frac{y}{y(1-y)} = \frac{1-\cancel{y}+\cancel{y}}{y(1-y)} = \frac{1}{y(1-y)} \quad \checkmark$$

$$\Rightarrow \int \frac{dy}{y} + \int \frac{dy}{1-y} = \int \lambda \, dt$$

$$\ln|y| - \ln|1-y| = \lambda t + C^*$$

$$\ln\left|\frac{y}{1-y}\right| = \lambda t + C^* \quad \Big| \, e^{(\dots)}$$

$$\frac{y}{1-y} = e^{\lambda t + C^*} = e^{\lambda t} \cdot \underbrace{e^{C^*}}_{=C}$$

$$y = (1-y)\cdot C \cdot e^{\lambda t}$$

$$y = Ce^{\lambda t} - yCe^{\lambda t}$$

$$y + yCe^{\lambda t} = Ce^{\lambda t}$$

$$y(1 + Ce^{\lambda t}) = Ce^{\lambda t}$$

$$y = \frac{Ce^{\lambda t}}{1 + Ce^{\lambda t}}$$

AB: $$y(0) = y_0 = \frac{C \cdot \overbrace{e^{0}}^{=1}}{1 + C \cdot \underbrace{e^{0}}_{=1}}$$

$$y_0 = \frac{C}{1+C}$$

$$y_0(1+C) = C$$

$$y_0 + y_0 C = C$$

$$y_0 = C - y_0 C$$

$$y_0 = C(1 - y_0)$$

$$C = \frac{y_0}{1-y_0}$$

$$\underline{\underline{y}} = \frac{\frac{y_0}{1-y_0} e^{\lambda t}}{1 + \frac{y_0}{1-y_0} e^{\lambda t}} = \frac{\frac{y_0}{\cancel{1-y_0}} \cdot e^{\lambda t}}{\frac{1-y_0}{\cancel{1-y_0}} + \frac{y_0}{\cancel{1-y_0}} \cdot e^{\lambda t}} =$$

$$= \frac{y_0 e^{\lambda t}}{1 - y_0 + y_0 e^{\lambda t}} = \frac{y_0 \cdot \cancel{e^{\lambda t}}}{\cancel{e^{\lambda t}} \cdot \left(\frac{1-y_0}{e^{\lambda t}} + y_0\right)} =$$

$$\underline{\underline{= \frac{y_0}{y_0 + (1-y_0) \cdot e^{-\lambda t}}}}$$

Lösung der logistischen DGL mittels Reihenansatz

$$\dot{y} = \lambda \cdot y \cdot (1-y) \quad \text{mit } y(0) = y_0$$

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \dots$$

$$\dot{y} = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \dots$$

$$\Rightarrow a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \dots = \lambda(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots) \cdot (1 - a_0 - a_1 t - a_2 t^2 - a_3 t^3 - \dots)$$

•) $y(0) = y_0 = a_0$

•) $a_1 = \lambda \cdot a_0 - \lambda a_0^2 = \lambda y_0 (1 - y_0)$

•) $2a_2 = \lambda a_1 - \lambda a_0 a_1 - \lambda a_0 a_1 =$

$$= \lambda \cdot \lambda y_0 (1-y_0) - \lambda y_0 \cdot \lambda y_0 (1-y_0) - \lambda y_0 \lambda y_0 (1-y_0) =$$

$$= \lambda^2 y_0 (1-y_0) \cdot (1 - y_0 - y_0) =$$

$$= \lambda^2 y_0 (1-y_0) \cdot (1 - 2y_0)$$

$$\Rightarrow a_2 = \frac{1}{2} \lambda^2 y_0 (1-y_0)(1-2y_0)$$

$$\underline{y = y_0 + \lambda y_0 (1-y_0) t + \frac{1}{2} \lambda^2 y_0 \cdot (1-y_0)(1-2y_0) \cdot t^2 + O(t^3)}$$

↑ ist ident mit der Taylorentwicklung der analytischen Lösung! ✓