Herleitung der Navier-Stokes-Gleichungen in 1D für kompressible Strömungen

1) **KG-Gleichung:** $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho \cdot v) = 0$ (2/4)

Keine Änderung zu Eulergleichungen, da Reibungskraft hier nicht eingeht.

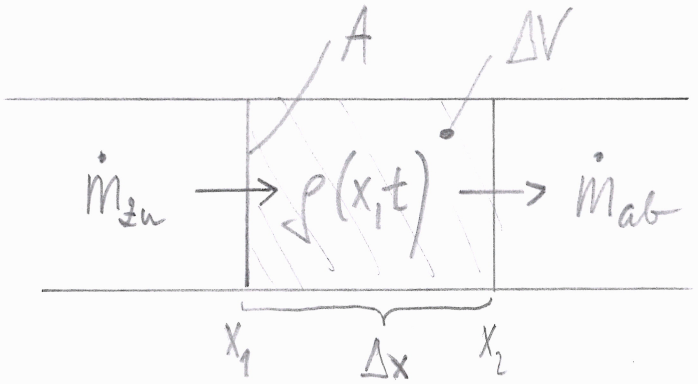
2) **Impulsgleichung:** siehe 3/4

3) **Energiegleichung:** siehe 4/4

4) **„Materialgesetz":** zB: „Gasgleichung"

$$\frac{p \cdot V}{T} = n \cdot R = \frac{p}{\rho \cdot T}$$

Die Kontinuitätsgleichung



$\Delta x: I_x = [x_1, x_2]$

$\Delta t: I_t = [t_1, t_2]$

Während dt: $dm_{zu} = \rho(x_1,t)\cdot A\cdot v(x_1,t)\,dt$, $dm_{ab} = \rho(x_2,t)\,A\,v(x_2,t)\,dt$

$$\dot m_{zu} = \dot m(x_1,t) = \rho(x_1,t)\cdot A\cdot v(x_1,t),\quad \dot m_{ab} = \dot m(x_2,t) = \rho(x_2,t)\cdot A\cdot v(x_2,t)$$

$$dm = dm_{zu} - dm_{ab} \Rightarrow \text{in } [t_1,t_2]\ \Delta m = \int_{t_1}^{t_2} dm = \int_{t_1}^{t_2}(dm_{zu} - dm_{ab})$$

$$dm = \left(\rho(x,t_2) - \rho(x,t_1)\right)\cdot A\cdot dx$$ ... Massenänderung im Intervall $[t_1,t_2]$ im Volumen $dV = A\cdot dx$

$\downarrow \int_{x_1}^{x_2}$

$$\Delta V: \begin{cases} \Delta m = \int_{x_1}^{x_2}\left(\rho(x,t_2) - \rho(x,t_1)\right)\cdot \overbrace{A\cdot dx}^{=dV} \quad \text{im Intervall } [t_1,t_2] \\ \Delta m = \int_{t_1}^{t_2}(\dot m_{zu} - \dot m_{ab})\,dt = \left(\int_{t_1}^{t_2}(dm_{zu} - dm_{ab})\right) = \end{cases}$$

im Zeitintervall $[t_1,t_2]$

$$= \int_{t_1}^{t_2}\left(\dot m(x_1,t) - \dot m(x_2,t)\right)dt = \int_{t_1}^{t_2}\left(\rho(x_1,t)\cdot v(x_1,t) - \rho(x_2,t)\cdot v(x_2,t)\right)\cdot A\cdot dt$$

$$\rightarrow \int_{x_1}^{x_2}\underbrace{\left(\rho(x,t_2) - \rho(x,t_1)\right)}_{=\int_{t_1}^{t_2}\frac{\partial\rho(x,t)}{\partial t}dt} A\cdot dx = -\int_{t_1}^{t_2}\underbrace{\left(\rho(x_2,t)\cdot v(x_2,t) - \rho(x_1,t)\,v(x_1,t)\right)}_{=\int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\rho(x,t)\cdot v(x,t)\right)dx}\cdot A\,dt$$

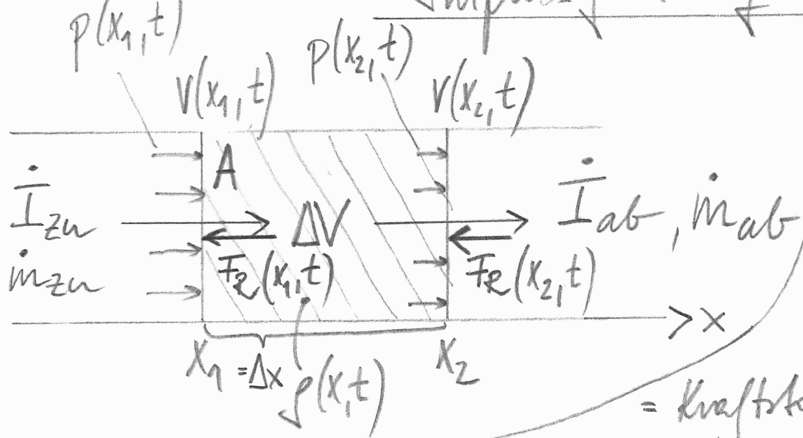
HS der Diff- & Int-Rechnung!

$$\Rightarrow \int_{x_1}^{x_2}\int_{t_1}^{t_2}\left(\frac{\partial\rho(x,t)}{\partial t}\cdot A + \frac{\partial}{\partial x}\left(\rho(x,t)\cdot v(x,t)\right)\cdot A\right)dt\,dx = 0,$$

$$\forall [t_1,t_2] \wedge \forall [x_1,x_2] \Rightarrow \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho\cdot v) = 0$$

Navier-Stokes-Gleichungen (1D)

Impulsgleichung



Im Intervall dt gilt:

$$dm_{zu} = \rho(x_1,t)\cdot A\cdot v(x_1,t)\cdot dt$$

$$dm_{ab} = \rho(x_2,t)\cdot \underbrace{A\cdot v(x_2,t)\cdot dt}_{=dV}$$

$$dI_{zu} = dm_{zu}\cdot v(x_1,t) + \underbrace{p(x_1,t)\cdot A\cdot dt}_{=\text{Kraftstoß aufgrund des Druckes}} - \overbrace{F_R(x_1,t)\cdot dt}^{=\text{Kraftstoß aufgrund der Reibung}}$$

Nach Stokes:

$$F_R = \eta\cdot A\cdot \frac{\partial v}{\partial x}$$

η = dyn. Viskosität

$$\Rightarrow dI_{zu} = \left(\rho(x_1,t)\cdot v^2(x_1,t) + p(x_1,t) - \eta\frac{\partial v}{\partial x}(x_1,t)\right)\cdot A\cdot dt$$

analog: $$dI_{ab} = \left(\rho(x_2,t)\cdot v^2(x_2,t) + p(x_2,t) - \eta\frac{\partial v}{\partial x}(x_2,t)\right)\cdot A\cdot dt$$

$$\Rightarrow dI = I_{zu} - I_{ab}$$

In $[t_1,t_2]$: $$\Delta I = \int_{t_1}^{t_2} dI = \int_{t_1}^{t_2}\left(dI_{zu} - dI_{ab}\right) =$$

$$= \int_{t_1}^{t_2}\left(\rho(x_1,t)v^2(x_1,t) - \rho(x_2,t)v^2(x_2,t) + p(x_1,t) - p(x_2,t) - \eta\frac{\partial v}{\partial x}(x_1,t) + \eta\frac{\partial v}{\partial x}(x_2,t)\right)\cdot A\cdot dt$$

Weiters gilt in $[t_1,t_2]$ im infinitesimalen Volumen $dV = A\cdot dx$:

$$dI_V = \rho(x,t_2)\cdot v(x,t_2)\cdot \underbrace{A\cdot dx}_{=dV} - \rho(x,t_1)\cdot v(x,t_1)\cdot A\cdot dx$$

$$\Rightarrow \int_{x_1}^{x_2}\left(\rho(x,t_2)\cdot v(x,t_2) - \rho(x,t_1)\cdot v(x,t_1)\right)\cdot A\cdot dx = \int_{t_1}^{t_2}\Big(\rho(x_1,t)\cdot v^2(x_1,t) - \rho(x_2,t)\cdot v^2(x_2,t) + p(x_1,t) - p(x_2,t)$$
$$- \eta\frac{\partial v}{\partial x}(x_1,t) + \eta\frac{\partial v}{\partial x}(x_2,t)\Big)\cdot A\cdot dt$$

Mittels HS der Diff & Int-Rechng:

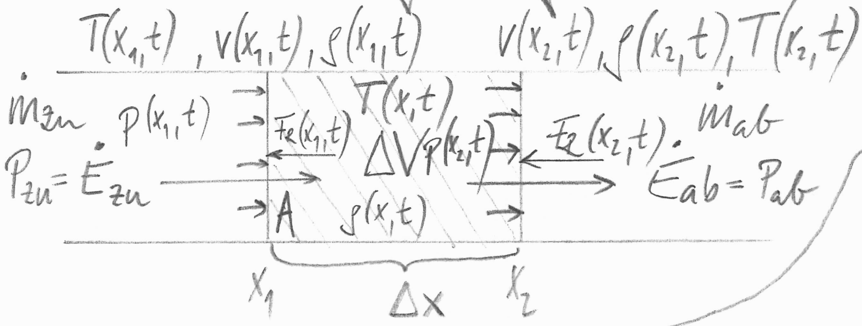
$$\int_{x_1}^{x_2}\int_{t_1}^{t_2}\frac{\partial(\rho\cdot v)}{\partial t}A\cdot dt\,dx = \int_{t_1}^{t_2}\underbrace{\int_{x_2}^{x_1}}_{\text{"}=-\int_{x_1}^{x_2}\text{"}}\frac{\partial}{\partial x}\left(\rho\cdot v^2 + p\right)\cdot A\cdot dx\,dt + \int_{t_1}^{t_2}\int_{x_1}^{x_2}\eta\frac{\partial^2 v}{\partial x^2}A\cdot dx\,dt$$

$$\Rightarrow \int_{x_1}^{x_2}\int_{t_1}^{t_2}\left(\frac{\partial(\rho v)}{\partial t} + \frac{\partial}{\partial x}\left(\rho v^2 + p\right) - \eta\frac{\partial^2 v}{\partial x^2}\right)\cdot A\cdot dt\,dx = 0$$

$$\Rightarrow \forall [x_1,x_2] \wedge [t_1,t_2]: \frac{\partial(\rho\cdot v)}{\partial t} + \frac{\partial}{\partial x}\left(\rho v^2 + p\right) - \eta\frac{\partial^2 v}{\partial x^2} = 0$$

mit $\rho = \rho(x,t)$, $v = v(x,t)$, $p = p(x,t)$

Navier-Stokes-Gleichungen, 1D, Energiegleichung



In dt gilt:

$$dm_{zu} = \rho(x_1,t)\cdot A\cdot v(x_1,t)\cdot dt$$

$$dm_{ab} = \rho(x_2,t)\cdot A\cdot v(x_2,t)\cdot dt$$

Stokes: $F_R = \eta\cdot A\cdot\frac{\partial v}{\partial x}$

$$dE_{zu} = \frac{1}{2}dm_{zu}\cdot v^2(x_1,t) + p(x_1,t)\cdot A\cdot dt - F_R(x_1,t)\cdot v(x_1,t)\cdot dt + c_V\cdot dm_{zu}\cdot T(x_1,t) =$$

$$= \frac{1}{2}\rho(x_1,t)\cdot A\cdot v^3(x_1,t)\,dt + p(x_1,t)\cdot A\cdot v(x_1,t)\,dt - \eta\cdot A\cdot\frac{\partial v}{\partial x}(x_1,t)\cdot v(x_1,t)\cdot dt + c_V\,\rho(x_1,t)\cdot A\cdot v(x_1,t)\cdot T(x_1,t)\cdot dt$$

dE_{ab} ... analog!

In $[t_1,t_2]$ gilt: $\Delta E = \int_{t_1}^{t_2}\left(dE_{zu} - dE_{ab}\right)$ $\boxed{1}$

Weiters gilt in dV während $[t_1,t_2]$:

$$dE_V = \frac{1}{2}\rho(x,t_2)\cdot A\cdot v^2(x,t_2)\,dx - \frac{1}{2}\rho(x,t_1)\cdot A\cdot v^2(x,t_1)\cdot dx + \left(c_V\cdot\rho(x,t_2)\cdot A\cdot T(x,t_2) - c_V\rho(x,t_1)T(x,t_1)\right)dx$$

$\Rightarrow$ In ΔV: $\Delta E = \int_{x_1}^{x_2} dE_V = \int_{x_1}^{x_2}\left(\frac{1}{2}\rho(x,t_2)\cdot v^2(x,t_2) - \frac{1}{2}\rho(x,t_1)\cdot v^2(x,t_1)\right)\cdot A\cdot dx$

$$+ \int_{x_1}^{x_2} c_V\left(\rho(x,t_2)\cdot T(x,t_2) - \rho(x,t_1)\cdot T(x,t_1)\right)\cdot A\cdot dx \quad \boxed{2}$$

$\boxed{2} = \boxed{1}$ $\Rightarrow$ Mittels HS d. Int-Diff-Rechnung:

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho(x,t)\cdot v^2(x,t) + c_V\rho(x,t)\cdot T(x,t)\right)\cdot A\cdot dt\cdot dx =$$

$$= -\int_{t_1}^{t_2}\int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\frac{1}{2}\rho(x,t)\cdot v^3(x,t) + p(x,t)v(x,t) - \eta\, v(x,t)\cdot\frac{\partial v}{\partial x}(x,t) + c_V\,\rho(x,t)\,v(x,t)\,T(x,t)\right)A\,dx\,dt$$

$$\Rightarrow \int_{x_1}^{x_2}\int_{t_1}^{t_2}\left(\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + c_V\rho T\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho v^3 + p\cdot v - \eta v\frac{\partial v}{\partial x} + c_V\rho v T\right)\right)\cdot A\cdot dx\cdot dt = 0$$

$$\forall [x_1,x_2]\ \&\ [t_1,t_2]:\ \frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2}\rho v^2 + c_V\rho T\right)}_{=\rho_E} + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho v^3 + p\cdot v - \eta\cdot v\frac{\partial v}{\partial x} + c_V\rho v\cdot T\right) = 0$$

$= \rho_E\cdot v$ mit ρ_E = „E-Dichte"

$$\Rightarrow \frac{\partial}{\partial t}\rho_E + \frac{\partial}{\partial x}\left(v\cdot\left(\rho_E + p - \eta\cdot\frac{\partial v}{\partial x}\right)\right) = 0$$

mit $p = p(x,t)$
$\rho = \rho(x,t)$
$v = v(x,t)$
$T = T(x,t)$