Herleitung der Wellengleichung für EM-Wellen im Vakuum

Maxwell-Gleichungen:
$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \qquad \rho \ldots \text{Ladungsdichte}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{j} \ldots \text{Stromdichte}$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Im Vakuum gilt: $\underbrace{\rho = 0}_{\nexists \text{ Ladungen}}$ und $\underbrace{\vec{j} = 0}_{\nexists \text{ Stromfluss}}$

$$\Rightarrow \text{I}: \vec{\nabla}\cdot\vec{E} = 0$$
$$\text{II}: \vec{\nabla}\cdot\vec{B} = 0$$
$$\text{III}: \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \xRightarrow{\vec{\nabla}\times} \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$
$$\text{IV}: \vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \xRightarrow{\vec{\nabla}\times} \vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\times\left(\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$
$$\text{NW}: \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \underbrace{\vec{\nabla}^2}_{=\Delta}\vec{A} \quad (\text{siehe Blatt 2})$$

$$\text{III)} \; \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{\overset{(\text{I})}{=}0}) - \Delta\vec{E} = -\frac{\partial}{\partial t}(\underbrace{\vec{\nabla}\times\vec{B}}_{\overset{(\text{IV})}{=}\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}}) \Rightarrow \underline{\underline{+\Delta\vec{E} = +\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}}}$$

$$\text{VI)} \; \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{B}}_{\overset{(\text{II})}{=}0}) - \Delta\vec{B} = \mu_0\varepsilon_0\frac{\partial}{\partial t}(\underbrace{\vec{\nabla}\times\vec{E}}_{\overset{(\text{III})}{=}-\frac{\partial\vec{B}}{\partial t}}) \Rightarrow \underline{\underline{+\Delta\vec{B} = +\mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}}}$$

$$\Rightarrow \boxed{\begin{aligned}\frac{\partial^2\vec{E}}{\partial t^2} &= \frac{1}{\mu_0\varepsilon_0}\cdot\Delta\vec{E}\\ \frac{\partial^2\vec{B}}{\partial t^2} &= \frac{1}{\mu_0\varepsilon_0}\cdot\Delta\vec{B}\end{aligned}} \quad \text{mit } c^2 = \frac{1}{\mu_0\varepsilon_0} \text{ für eine fortschreitende Welle}$$

Nebenrechnung (= Graßmann Identität)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \left[\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \right] =$$

$$= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 A_y}{\partial x \partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial x \partial z} \\ \frac{\cdots}{\cdots} - \underline{\qquad} - \underline{\qquad} + \underline{\qquad} \\ \frac{\cdots}{\cdots} - \underline{\qquad} - \underline{\qquad} + \underline{\qquad} \end{pmatrix}$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} =$$

$$= \begin{pmatrix} \cancel{\frac{\partial^2 A_x}{\partial x^2}} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z} - \cancel{\frac{\partial^2 A_x}{\partial x^2}} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} \\ \cdots \\ \cdots \end{pmatrix}$$