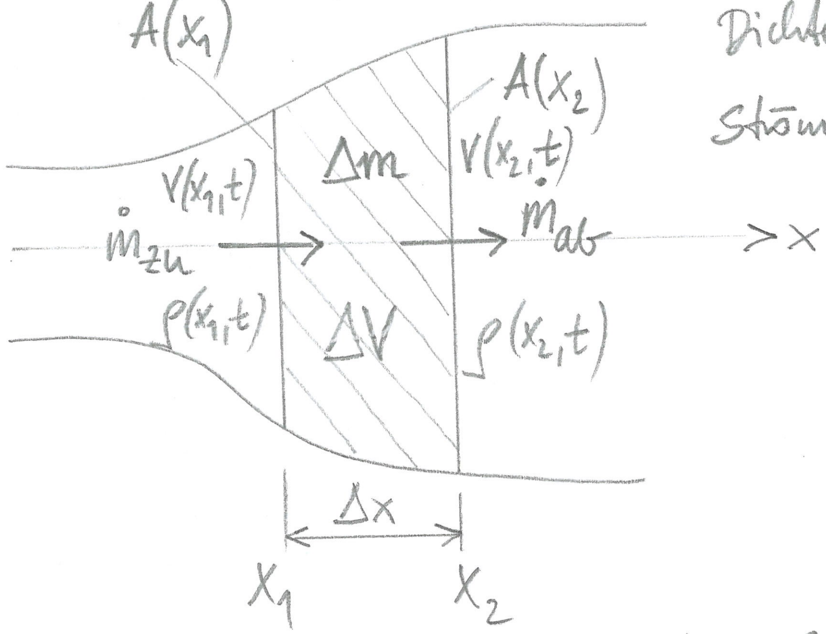
1D-Kontinuitätsgleichung für variable Querschnittsfläche A



Dichte: $\rho = \rho(x,t)$

Strömungsgeschw.: $v = v(x,t)$

Innerhalb von dt gilt an den Stelle x_1 & x_2:

$$dm_{zu} = \rho(x_1,t)\cdot v(x_1,t)\cdot A(x_1)\cdot dt$$

$$dm_{ab} = \rho(x_2,t)\cdot v(x_2,t)\cdot A(x_2)\cdot dt$$

Innerhalb von $[t_1,t_2]$ gilt im Abschnitt dx:

$$dm = \left(\rho(x,t_2)\cdot A(x) - \rho(x,t_1)\cdot A(x)\right) dx$$

Gesamtänderung in ΔV innerhalb von $[t_1,t_2]$:

$$\Delta m = \int_{x_1}^{x_2} dm = \int_{x_1}^{x_2} \underbrace{\left(\rho(x,t_2) - \rho(x,t_1)\right)}_{=\int_{t_1}^{t_2} \frac{\partial \rho(x,t)}{\partial t}\, dt} \cdot A(x)\, dx = \int_{x_1}^{x_2}\int_{t_1}^{t_2} \frac{\partial \rho(x,t)}{\partial t} A(x)\, dt\, dx$$

und

$$\Delta m = \int_{t_1}^{t_2} (dm_{zu} - dm_{ab}) =$$

$$= \int_{t_1}^{t_2} \underbrace{\left(\rho(x_1,t)\, v(x_1,t)\cdot A(x_1) - \rho(x_2,t)\, v(x_2,t)\cdot A(x_2)\right)}\, dt$$

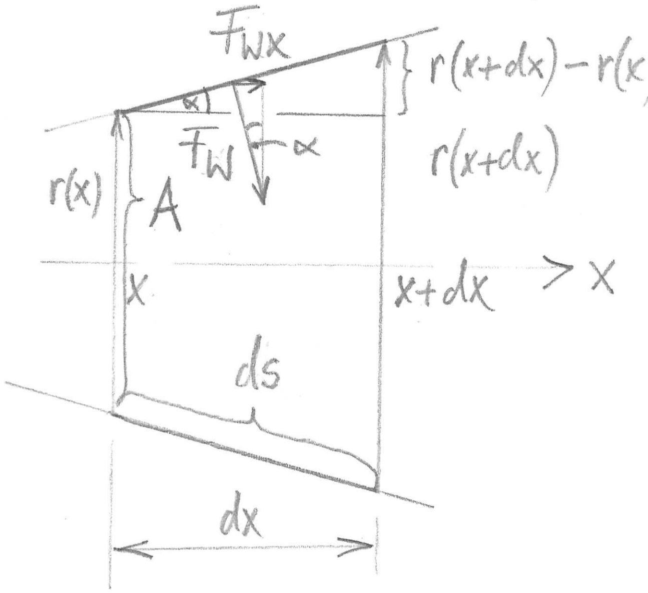
$$= \int_{t_1}^{t_2} \int_{x_2}^{x_1} \frac{\partial}{\partial x}\left(\rho(x,t)\, v(x,t)\cdot A(x)\right) dx\, dt$$

$$\Rightarrow \int_{x_1}^{x_2}\int_{t_1}^{t_2} A(x)\,\frac{\partial \rho(x,t)}{\partial t}\,dt\,dx = -\int_{x_1}^{x_2}\int_{t_1}^{t_2} \frac{\partial}{\partial x}\left(\rho(x,t)\,v(x,t)\,A(x)\right) dt\,dx$$

... muss $\forall\,[x_1,x_2]$ & $\forall\,[t_1,t_2]$ gelten

$$\Rightarrow \quad \underline{\underline{A(x)\cdot\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x}\left(\rho(x,t)\,v(x,t)\cdot A(x)\right) = 0}}$$

Vorprogramm zur Herleitung der Impulsglg bei variabler Querschnittsfläche A



$r(x+dx)-r(x) \approx \cancel{r(x)} + dr - \cancel{r(x)} = dr$

$$\cos\alpha = \frac{dx}{ds} \quad\Downarrow\quad ds = \frac{dx}{\cos\alpha} \qquad \Bigg| \qquad \tan\alpha = \frac{dr}{dx}$$

Fläche des infinitesimalen Kegelstumpf - Es (Mantelfl.):

$$dA_{KS} = 2r\pi \cdot ds = 2\pi r \cdot \frac{dx}{\cos\alpha}$$

$$\sin\alpha = \frac{F_{Wx}}{F_W}, \quad \text{mit } F_d = p \cdot dA_{KS} = p \cdot 2r\pi \cdot \frac{dx}{\cos\alpha}$$

$$\Rightarrow F_{Wx} = F_W \cdot \sin\alpha = p \cdot 2r\pi \cdot \frac{dx}{\cos\alpha} \cdot \sin\alpha =$$

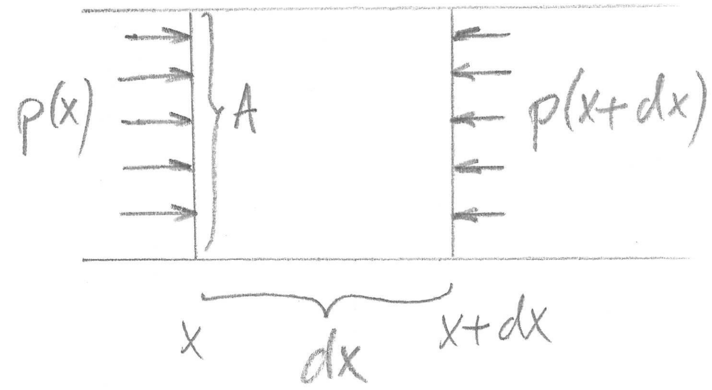
$$= p \cdot 2r\pi \cdot dx \cdot \underbrace{\frac{\sin\alpha}{\cos\alpha}}_{=\tan\alpha} =$$

$$= p \cdot 2r\pi \cdot \cancel{dx}\,\frac{dr}{\cancel{dx}} = p \cdot \underbrace{2r\pi\, dr}$$

WW: $A = r^2\pi \Rightarrow dA = 2r\pi\, dr$

$$\Rightarrow F_{Wx} = p \cdot dA$$

Kräftebilanz Druckkraft bei <u>konst.</u> Querschnittsfläche A

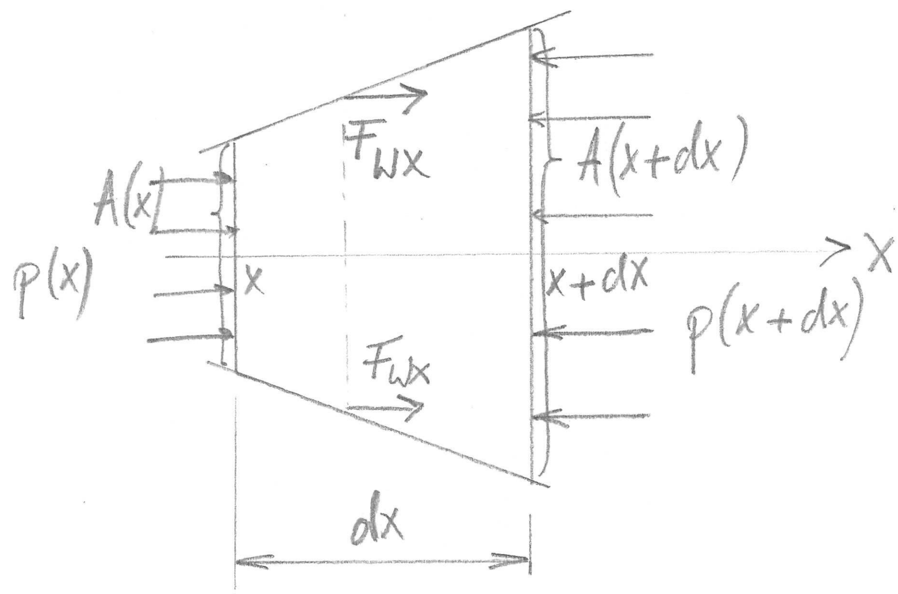


$$\underline{\underline{dF_{geo}}} = p(x)\cdot A - p(x+dx)\cdot A =$$

$$= A\cdot\left(\cancel{p(x)} - \cancel{p(x)} - \frac{dp(x)}{dx}\,dx\right) =$$

$$\underline{\underline{= -A\,\frac{dp(x)}{dx}\cdot dx}}$$

Kräftebilanz Druckkraft bei variabler Querschnittsfläche A



WW: $F_{Wx} = p \cdot dA$

$$dF_{ges} = p(x) \cdot A(x) - \underbrace{p(x+dx) \cdot A(x+dx)}_{\text{TAYLOR}} + p(x) \cdot dA(x)$$

$$= p(x) \cdot A(x) - \Big(p(x) \cdot A(x) + \underbrace{\frac{d(p \cdot A)}{dx}}_{\text{Produktregel}} dx \Big) + p(x) \cdot \frac{dA(x)}{dx} \cdot dx =$$

$$= \cancel{p(x) \cdot A(x)} - \cancel{p(x) \cdot A(x)} - \frac{dp(x)}{dx} \cdot A(x)\, dx \underbrace{- \cancel{p(x) \frac{dA(x)}{dx} dx} + \cancel{p(x) \cdot \frac{dA(x)}{dx} dx}} =$$

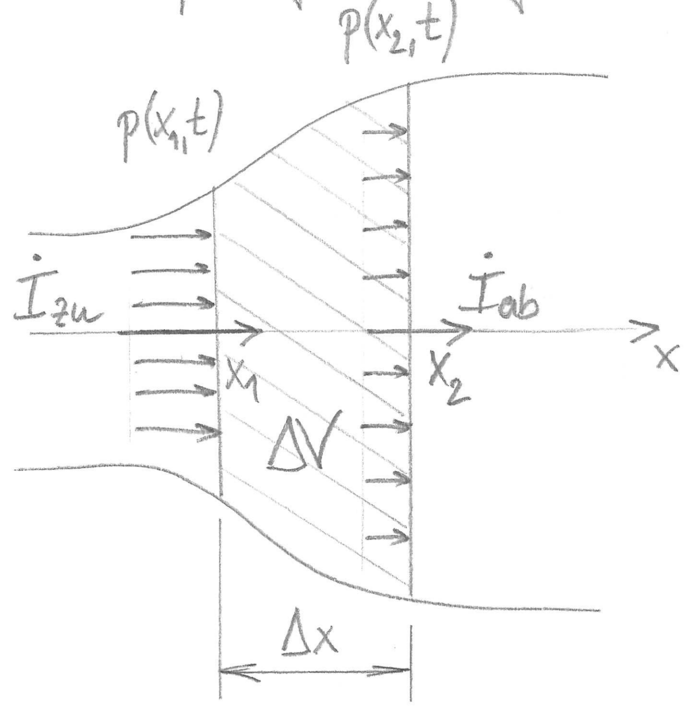
$$dF_{ges} = - A(x) \frac{dp(x)}{dx} \cdot dx$$

Ein Vergleich zeigt, dass die Gesamtkraft ident mit der Gesamtkraft bei konst. Querschnittsfl. A ist.

Begr: Der zusätzl. Term wird durch den 2. Term in der Produktregel „kompensiert".

⟹ Der variable Rohrdurchmesser hat trotzdem einen Einfluss auf die Impulsgleichung!

1D-Impulsgleichung für variable Querschnittsfläche A



Innerhalb von dt gilt an den Intervallgrenzen $[x_1, x_2]$:

$$\dot{m}_{zu} = \rho(x_1,t)\cdot A(x_1)\cdot v(x_1,t)$$

$$\dot{m}_{ab} = \rho(x_2,t)\cdot A(x_2)\cdot v(x_2,t)$$

bzw.

$$\dot{I}_{zu} = \dot{m}_{zu}\, v(x_1,t) + p(x_1,t)\cdot A(x_1)$$

$$\dot{I}_{ab} = \dot{m}_{ab}\cdot v(x_2,t) + p(x_2,t)\cdot A(x_2)$$

Innerhalb von $[t_1, t_2]$ gilt im Abschnitt dx:

$$dI = \left(\rho(x,t_2)A(x)v(x,t_2) - \rho(x,t_1)A(x)v(x,t_1)\right)dx \underbrace{- \int_{t_1}^{t_2} p(x,t)\,\frac{\partial A}{\partial x}\,dx\,dt}_{=\text{Wandbeitrag}}$$

$\Rightarrow$ ΔI innerhalb von $[t_1, t_2]$ in ΔV:

$$\Delta I = \int_{t_1}^{t_2}\left(\dot{I}_{zu} - \dot{I}_{ab}\right)dt = \int_{x_1}^{x_2} dI$$

$$\int_{t_1}^{t_2}\left(\rho(x_1,t)A(x_1)v^2(x_1,t) + p(x_1)A(x_1) - \rho(x_2,t)A(x_2)v^2(x_2,t) - p(x_2,t)A(x_2)\right)dt =$$

$$= \int_{x_1}^{x_2}\left(\rho(x,t_2)A(x)\,v(x,t_2) - \rho(x,t_1)A(x)v(x,t_1)\right)dx - \int_{x_1}^{x_2}\int_{t_1}^{t_2} p(x,t)\cdot\frac{\partial A(x)}{\partial x}\,dt\,dx$$

$\Rightarrow$ mit dem HS der Diff.- & Int.-Rechnung:

$$-\int_{t_1}^{t_2}\int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\rho(x,t)A(x)\,v^2(x,t) + p(x,t)\cdot A(x)\right)dx\,dt =$$

$$= \int_{x_1}^{x_2}\int_{t_1}^{t_2}\frac{\partial}{\partial t}\left(\rho(x,t)A(x)\,v(x,t)\right)dt\,dx - \int_{x_1}^{x_2}\int_{t_1}^{t_2} p(x,t)\,\frac{\partial A(x)}{\partial x}\,dt\,dx$$

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2}\left[\frac{\partial}{\partial t}(\rho A v)+\frac{\partial}{\partial x}(\rho A v^2+p\cdot A)-p\frac{\partial A}{\partial x}\right]dt\,dx=0$$

... muss $\forall[t_1,t_2] \wedge \forall[x_1,x_2]$ gelten:

$$\Longrightarrow \quad \frac{\partial}{\partial t}(\rho A v)+\frac{\partial}{\partial x}(\rho A v^2+p A)-p\frac{\partial A}{\partial x}=0$$

$A=A(x)$

$$A\frac{\partial(\rho v)}{\partial t}+\frac{\partial}{\partial x}(\rho A v^2)+\frac{\partial p}{\partial x}A+\underbrace{p\frac{\partial A}{\partial x}-p\frac{\partial A}{\partial x}}_{=0}=0$$

$$\underline{\underline{A\frac{\partial(\rho v)}{\partial t}+\frac{\partial}{\partial x}(\rho A v^2)+A\frac{\partial p}{\partial x}=0}}$$

Alternative Form der Impulsglg

$$\underline{KG:}\quad A\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}vA + \rho\frac{\partial v}{\partial x}A + \rho v\frac{\partial A}{\partial x} = 0 \quad \text{(Produktregel)}$$

$$\underline{IG:}\quad \underbrace{A\frac{\partial\rho}{\partial t}v}_{1} + A\rho\frac{\partial v}{\partial t} + \underbrace{\frac{\partial\rho}{\partial x}Av^2}_{2} + \underbrace{\rho\frac{\partial A}{\partial x}v^2}_{4} + \underbrace{\rho A\cdot 2v\frac{\partial v}{\partial x}}_{3} + A\frac{\partial p}{\partial x} = 0$$

(Produkt- & Kettenregel)

$$\underbrace{v\cdot\left(\underbrace{A\frac{\partial\rho}{\partial t}}_{1} + \underbrace{\frac{\partial\rho}{\partial x}\cdot v\cdot A}_{2} + \underbrace{\rho\frac{\partial v}{\partial x}\cdot A}_{3a} + \underbrace{\rho v\frac{\partial A}{\partial x}}_{4}\right)}_{=0,\ KG} + \underbrace{\rho A v\frac{\partial v}{\partial x}}_{3b} \oplus$$

$$\oplus\ A\rho\frac{\partial v}{\partial t} + \frac{\partial p}{\partial x}A = 0$$

$$\Rightarrow\quad \rho\cancel{A}v\frac{\partial v}{\partial x} + \cancel{A}\rho\frac{\partial v}{\partial t} + \cancel{A}\frac{\partial p}{\partial x} = 0$$

$$\underline{\underline{\rho v\frac{\partial v}{\partial x} + \rho\frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} = 0}}$$

Herleitung der Webster Horn-Gleichung

WW: KG: $A \cdot \frac{\partial \rho_{ges}}{\partial t} + \frac{\partial}{\partial x}(\rho_{ges} v A) = 0$

alternat. IG: $\rho_{ges} v \cdot \frac{\partial v}{\partial x} + \rho_{ges} \frac{\partial v}{\partial t} + \frac{\partial p_{ges}}{\partial x} = 0$

Materialgesetz: aus der adiabatischen Zustandsänderung folgt: $p = \chi \cdot R_S \cdot T \cdot \rho$ (Herleitung siehe Wellengl. für Schall 1D) für eine fortschreitende Welle gilt

$c^2 = \chi \cdot R_S \cdot T \Rightarrow p = c^2 \cdot \rho$

$\rho_{ges} = \rho_0 + \rho$ $\qquad \rho = \rho(x,t)$, $p = p(x,t)$, $A = A(x)$

$p_{ges} = p_0 + p$

KG: $A \frac{\partial}{\partial t}(\rho_0 + \rho) + \frac{\partial}{\partial x}((\rho_0 + \rho) v A) = 0$

$$A \underbrace{\frac{\partial \rho_0}{\partial t}}_{=0} + A \frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial}{\partial x}(vA) + \underbrace{\frac{\partial}{\partial x}(\rho v A)}_{=0 \text{ ... Linearisierung}} = 0$$

a) $A \frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial}{\partial x}(v \cdot A) = 0 \Rightarrow A \frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial v}{\partial x} A + \rho_0 v \frac{\partial A}{\partial x} = 0$

IG: $\underbrace{(\rho_0 + \rho)}_{\approx \rho_0} \underbrace{v \cdot \frac{\partial v}{\partial x}}_{\approx 0} + \underbrace{(\rho_0 + \rho)}_{\approx \rho_0} \frac{\partial v}{\partial t} + \underbrace{\frac{\partial}{\partial x}(p_0 + p)}_{=0} = 0$ $\qquad \frac{\partial}{\partial t}$

Linearisierung

b) $\rho_0 \frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} = 0 \quad \left| \frac{\partial}{\partial x} \right| \cdot A$

$$\left.\begin{aligned} &\text{I: } \cancel{\rho_0 A \frac{\partial^2 v}{\partial t \partial x}} + A \frac{\partial^2 p}{\partial x^2} = 0 \\ &\text{II: } A \frac{\partial^2 \rho}{\partial t^2} + \cancel{\rho_0 A \frac{\partial^2 v}{\partial t \partial x}} + \rho_0 \frac{\partial}{\partial t}\left(v \cdot \frac{\partial A}{\partial x}\right) = 0 \end{aligned}\right\} \text{II} - \text{I}$$

$$A\cdot\frac{\partial^2\rho}{\partial t^2} - A\cdot\frac{\partial^2 p}{\partial x^2} + \rho_0\,\frac{\partial}{\partial t}\left(v\cdot\frac{\partial A}{\partial x}\right) = 0$$

$A = A(x)$, also unabh. von t

$$A\,\underbrace{\frac{\partial^2\rho}{\partial t^2}}_{p = c^2\cdot\rho} - A\,\frac{\partial^2 p}{\partial x^2} + \cancel{\rho_0}\,\underbrace{\frac{\partial v}{\partial t}}_{= -\frac{1}{\rho_0}\cdot\frac{\partial p}{\partial x}\ (IG)}\cdot\frac{\partial A}{\partial x}$$

$$A\,\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - A\,\frac{\partial^2 p}{\partial x^2} - \frac{\partial p}{\partial x}\cdot\frac{\partial A}{\partial x} = 0$$

$$A\,\frac{1}{c^2}\,\frac{\partial^2 p}{\partial t^2} = A\,\frac{\partial^2 p}{\partial x^2} + \frac{\partial p}{\partial x}\,\frac{\partial A}{\partial x}$$

Produktregel rückwärts

$$A\cdot\frac{1}{c^2}\,\frac{\partial^2 p}{\partial t^2} = \frac{\partial}{\partial x}\left(A\cdot\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial^2 p}{\partial t^2} = c^2\cdot\frac{1}{A}\,\frac{\partial}{\partial x}\left(A\cdot\frac{\partial p}{\partial x}\right) \quad \text{mit } p = p(x,t) \wedge A = A(x)$$

Das geht ausgehend von a) & b) auch einfacher:

a) $A \frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial}{\partial x}(v \cdot A) = 0 \quad \Big| \frac{\partial}{\partial t}$

b) $\rho_0 \frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} = 0$

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a) $A \frac{\partial^2 \rho}{\partial t^2} + \rho_0 \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} \cdot A\right) = 0$, da A unabh. von t

b) $\frac{\partial v}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x}$

$$A \frac{\partial^2 \rho}{\partial t^2} + \cancel{\rho_0} \cdot \frac{\partial}{\partial x}\left(-\frac{1}{\cancel{\rho_0}} \cdot \frac{\partial p}{\partial x} \cdot A\right) = 0$$

$\rho = \frac{1}{c^2} \cdot p$

$$A \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \frac{\partial}{\partial x}\left(A \frac{\partial p}{\partial x}\right)$$

$$\frac{\partial^2 p}{\partial t^2} = c^2 \cdot \frac{1}{A} \frac{\partial}{\partial x}\left(A \frac{\partial p}{\partial x}\right)$$